MA 241 : Ordinary Differential Equations (JAN-APR, 2018)

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Problem set 1

1. Consider the initial value problem

$$\frac{dy}{dt} = ay(t) - by^2(t),\ y(t_0) = y_0$$

where $a, b > 0$, $t_0, y_0 \in \mathbb{R}$. Assume the unique existence of the (local) solution $y = y(t)$ in the interval (t_1, t_2) where $t_0 \in (t_1, t_2)$.

a. Without solving the problem, show that y satisfies
i) $y(t) > \frac{a}{b}$ if $y_0 > \frac{a}{b}$; ii) $0 < y(t) < \frac{a}{b}$ if $0 < y_0 < \frac{a}{b}$; iii) $y(t) < 0$ if $y_0 < 0$.

b. Now solve the problem to get the implicit form

$$log \frac{|y|}{y_0} \frac{|a - by_0|}{|a - by|} = t - t_0.$$

c. Use (a) to get the explicit form

$$y(t) = \frac{ay_0}{by_0 + (a - by_0)e^{-a(t-t_0)}}$$

d. In each of the three cases as in (a), find the maximal interval of existence (t_*, t^*), that is the minimal t_* and the maximal t^* so that y given in (c) is the solution of the initial value problem. (Note that t^* can be $+\infty$ or t_* can be $-\infty$). Further, find

$$\lim_{t \uparrow t_*} y(t) \quad \text{and} \quad \lim_{t \downarrow t^*} y(t).$$

e. In each of cases, find $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ and analyze the shape of the curve.

f. In which case the global solution exists, that is $t_* = -\infty$ and $t^* = +\infty$.

g. Sketch the graphs and then plot it with various initial values using any software.

h. Let $y = y(t)$ be the solution as above and $z = z(t)$ be the solution to the initial value problem:

$$\frac{dz}{dt} = az(t) - bz^2(t),\ z(t_1) = y_0.$$

Represent z in terms of y. Sketch with different initial times. Do you observe any property? Can you write the observed property for the general problem

$$\frac{dy}{dt} = f(y(t)),\ y(t_0) = y_0.$$

2. Consider the linear model of the atomic waste disposal problem:

$$\frac{dV}{dt} + \frac{cg}{W}V(t) = \frac{g}{W}(W - B),\ V(0) = 0,$$

where $V = V(t)$ is the velocity at the time t.
a. Find the solution V and find the limit $\lim_{t \to \infty} V(t)$.

b. Now derive the non-linear model:

$$\frac{v}{W - B - cv} \frac{dv(y)}{dy} = \frac{g}{W},\ v(0) = 0$$

where $v = v(y)$ is the velocity at the distance y.
c. Solve the equation to get the solution:

$$\frac{gy}{W} = -\frac{v}{c} - \frac{W - B}{C^2} log \frac{W - B - cv}{W - B}.$$